

The background is a vibrant, abstract collage. It features a large, glowing yellow sphere on the left, several smaller blue and orange spheres, and white orbital paths. In the top right corner, there are mathematical formulas:  $\sqrt{5}$ ,  $\left(\frac{1-\sqrt{5}}{2}\right)^n$ , and  $2$ . A large, dark, stylized letter 'L' is also visible on the left side.

# Math 1552

## ***Section 4.5: L'Hopital's Rule***

Math 1552 lecture slides adapted from the course materials  
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Options for review session  
time next Wednesday:

- 4:30-6:30 PM

- 5-7 PM

- 6-8 PM

- 7-9 PM

→ Fill out the  
TP poll  
today by  
12:45



# Indeterminate Forms

$$\frac{0}{0}, \frac{\infty}{\infty}$$

→ Can directly apply L'Hopital's rule

$$1^{\infty}, 0^0, \infty^0$$

$$0 \cdot \infty, \infty - \infty$$

→ looking to evaluate limits with an initial indeterminate form

# L'Hopital's Rule

Let  $f$  and  $g$  be two functions. Then IF:

a)  $f$  and  $g$  are differentiable,

b)  $\frac{f(x)}{g(x)}$  has the indeterminate form of

$$\frac{0}{0} \quad \text{OR} \quad \frac{\infty}{\infty}$$

c)  $\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} = L$  exists

important, cannot apply the rule without one of these forms

THEN:

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} = L$$

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{3^x - 1}{4^x - 1}$$

A. 0

B. 1

C.  $\ln(3/4)$

D.  $(\ln 3)/(\ln 4)$

$$3^0 - 1 = 1 - 1 = 0$$

$$4^0 - 1 = 1 - 1 = 0$$

0/0  $\rightarrow$  apply the rule

$$3^x = e^{x \cdot \ln(3)}$$

$$4^x = e^{x \cdot \ln(4)}$$

$$\frac{d}{dx} [3^x - 1]$$

for any value  $C$  over the reals:

$$e^{\ln(C)} = C$$

$$\ln(e^C) = C$$

$$= \frac{d}{dx} [e^{x \ln(3)} - 1]$$

$$= \ln(3) \cdot 3^x$$

$$\underline{\frac{d}{dx} [4^x - 1]} = \ln(4) \cdot 4^x$$

$$\text{So } \lim_{x \rightarrow 0} \frac{3^x - 1}{4^x - 1} = \lim_{x \rightarrow 0} \frac{\ln(3) \cdot 3^x}{\ln(4) \cdot 4^x}$$

(Now can evaluate the limit directly)

$$= \frac{\ln(3)}{\ln(4)}$$

$$\uparrow \frac{3^0}{4^0} = \frac{1}{1}$$

Example 2.1: Use L'Hopital's rule and logarithms to evaluate the following limit.

$$L = \lim_{x \rightarrow 0^+} \underbrace{x^{\frac{1}{\ln(5x)}}}_{f(x)} \quad 0^{-\infty}?$$

Logarithm rule:  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} e^{\ln(f(x))} = \exp\left(\lim_{x \rightarrow a} \ln(f(x))\right)$

$$L = \exp\left(\lim_{x \rightarrow 0^+} \frac{1}{\ln(5x)} \cdot \ln(x)\right)$$

$$\ln(L) = \lim_{x \rightarrow 0^+} \frac{\ln(x) \quad f(x)}{\ln(5x) \quad g(x)} \quad \left| \begin{array}{l} -\infty \\ -\infty \end{array} \right.$$

$$\exp(x) \equiv e^x$$

→ apply the rule



$$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\frac{1}{5x} \cdot 0.5}$$

$$= \lim_{x \rightarrow 0^+} 1 = 1$$

$$\ln(L) = 1 \Rightarrow L = e$$

Example 2.2: Use L'Hopital's rule and logarithms to evaluate the following limit.

(know this limit)

Logarithm rule:

$$L = \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x, \text{ } a \text{ is any constant } \infty$$
$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} e^{\ln(f(x))} = \exp\left(\lim_{x \rightarrow a} \ln(f(x))\right)$$

$$L = \exp\left(\lim_{x \rightarrow \infty} x \cdot \ln\left(1 + \frac{a}{x}\right)\right)$$

$$\hookrightarrow \ln(L) = \lim_{x \rightarrow \infty} x \cdot \ln\left(1 + \frac{a}{x}\right)$$
$$\rightarrow \infty \cdot \ln(1) \rightarrow \infty \cdot 0$$

→ how to apply L'Hopital's rule?

$$\ln(L) = \lim_{x \rightarrow \infty} \frac{\ln(1 + \frac{a}{x})}{\frac{1}{x}} \quad \frac{0}{0} \checkmark$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{a}{x}} \cdot \left(-\frac{a}{x^2}\right)}{\frac{-1}{x^2}}$$

$$\begin{aligned}\frac{d}{dx} \left[ \ln \left( 1 + \frac{a}{x} \right) \right] &= \frac{1}{1 + \frac{a}{x}} \cdot \frac{d}{dx} \left[ 1 + \frac{a}{x} \right] \\ &= \frac{1}{1 + \frac{a}{x}} \cdot \left( \frac{-a}{x^2} \right)\end{aligned}$$

$$= \lim_{x \rightarrow \infty} \frac{a}{1 + \frac{a}{x}} = a$$

$\xrightarrow{\quad} 0$

$$\rightarrow \ln(L) = a \Rightarrow L = e^a$$



What about this limit?

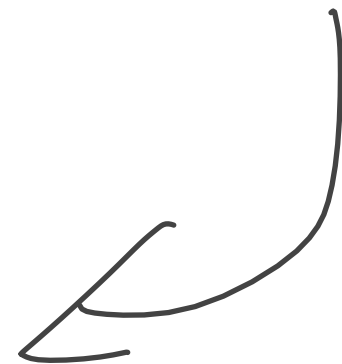
$$L_1 = \lim_{x \rightarrow \infty} \left( 1 + \frac{a}{x^2} \right)^{x^2}$$
$$\left( 1 + \frac{2}{x} + \frac{4}{x^2} \right)^{x^2}$$

$\rightarrow e^a$   
(work out the details on your own)

$$L_2 = \lim_{N \rightarrow 0} (1 + bN)^{\frac{1}{N}}$$

( $b=2$  on the next slide)

$$\rightarrow e^b$$



Evaluate the limit:

$$L = \lim_{x \rightarrow 0^+} (1 + 2x)^{\frac{1}{x}}$$

A.  $e^2$

B.  $e^{1/2}$

C. 1

D. Infinity

$$\ln(L) = \lim_{x \rightarrow 0^+} \frac{\ln(1+2x)}{x} \quad \frac{0}{0} \checkmark$$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{2}{1+2x}}{1}$$

$$= \lim_{x \rightarrow 0^+} \frac{2}{1+2x} = 2$$

$$\text{So } \ln(L) = 2 \iff L = e^2$$

# Compendia of Common Limits (memorize)

1) If  $x > 0$ , then  $\lim_{n \rightarrow \infty} x^{1/n} = 1$ .

2) If  $|x| < 1$ , then  $\lim_{n \rightarrow \infty} x^n = 0$ . ex:  $\lim_{N \rightarrow \infty} \left(\frac{1}{2}\right)^N = 0$  ( $x = \frac{1}{2}$ )

3) If  $\alpha > 0$ , then  $\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} = 0$ . ex:  $\lim_{N \rightarrow \infty} \frac{1}{N^2} = 0$  ( $\alpha = 2$ )

4)  $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$

5)  $\lim_{n \rightarrow \infty} \frac{\ln(n)}{n} = 0$

6)  $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$       7)  $\lim_{n \rightarrow \infty} n^{1/n} = 1$

from the previous example

What is the value of the following limit:

$$L = \lim_{N \rightarrow \infty} (2N)^{1/N}$$

$$= \lim_{N \rightarrow \infty} 2^{\frac{1}{N}} \cdot N^{1/N} = 1$$

$\xrightarrow{\text{by (7)}} 1$

$\xleftarrow{\text{by (1)}}$



Extra Problem I: Evaluate the following limit:

$$L = \lim_{w \rightarrow -6} \frac{\sin(2\pi w)}{w^2 - 36}$$

→ work this problem out on your own

→ solution:  $L = -\frac{\pi}{6}$

Extra Problem II: Evaluate the following limit:

$$L = \lim_{x \rightarrow 0^+} \frac{\sin(2x)}{2x}, \quad \lim_{x \rightarrow 0^+} \frac{\sin(ax)}{ax} = 1$$

→ work this problem out on  
your own

→ solution:  $L = 1$

→ compare to the familiar  
limit:  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

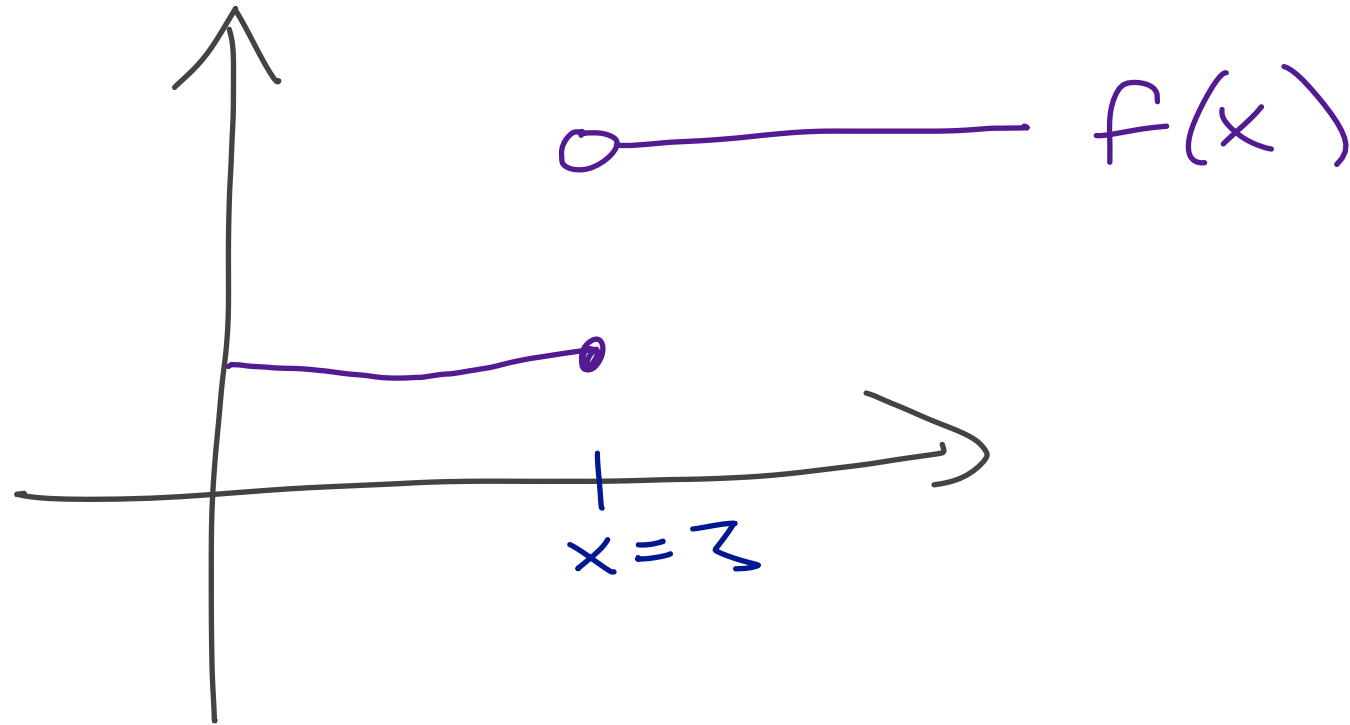
Extra Problem III: Evaluate the following limit:

$$L = \lim_{x \rightarrow \frac{1}{2}^+} \left( x - \frac{1}{2} \right) \tan(\pi x)$$

→ work this out on your own

→ solution:  $L = -\frac{1}{\pi}$

Examples of when limits do not exist:



$$\lim_{x \rightarrow 3} f(x) \text{ DNE}$$

## Bonus Practice Problems: Evaluate each of the following limits:

(In class: practice verifying that we get an indeterminate form in each case)

►  $\lim_{x \rightarrow \infty} \frac{x^2 - 2}{2x^2 - 3x + 1}$   $\frac{\infty}{\infty}$  **Hint:** Multiply through by  $1 = \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$ , and then take limits

►  $\lim_{t \rightarrow +\infty} \left[ t \cdot \ln \left( 1 + \frac{8}{t} \right) \right]$   $\infty \cdot 0 \rightarrow$  transform to apply the rule:

$\lim_{t \rightarrow +\infty} \frac{\ln \left( 1 + \frac{8}{t} \right)}{1/t} \quad \frac{0}{0}$

**Hint:** Multiply through by the conjugate

$1 = \frac{\sqrt{x^2 + 2} + \sqrt{x + 2}}{\sqrt{x^2 + 2} + \sqrt{x + 2}}$ , to simplify the numerator first

3 ►  $\lim_{x \rightarrow 0^+} \frac{3^x - 4^x}{x^2 - 2x}$

4 ►  $\lim_{x \rightarrow +\infty} \left[ \sqrt{x^2 + 2} - \sqrt{x + 2} \right]$

5 ►  $\lim_{x \rightarrow 0} x^{3x}$



$$\lim_{x \rightarrow 0^+} \frac{3^x - 4^x}{x^2 - 2x}$$

$\frac{0}{0} \rightarrow$  apply the rule directly

$$\lim_{x \rightarrow +\infty} \left[ \sqrt{x^2 + 2} - \sqrt{x + 2} \right]$$

$\infty - \infty$

Simplify by writing:

$$\frac{(\sqrt{x^2 + 2} - \sqrt{x + 2}) \cdot (\sqrt{x^2 + 2} + \sqrt{x + 2})}{(\sqrt{x^2 + 2} + \sqrt{x + 2})}$$

$$= \frac{(x^2 + 2 - (x + 2))}{\sqrt{x^2 + 2} + \sqrt{x + 2}} = \frac{(x^2 - x)}{\sqrt{x^2 + 2} + \sqrt{x + 2}}$$

So as  $x \rightarrow +\infty$ :  $\frac{\infty}{\infty}$

$0 \cdot (-\infty) \times$

$$L = \lim_{x \rightarrow 0^+} x^{3x}$$

$0^0$

$$\begin{aligned} \ln(L) &= \lim_{x \rightarrow 0^+} 3x \cdot \ln(x) \\ &= \lim_{x \rightarrow 0^+} \frac{\ln(x)}{\frac{1}{3x}} \quad \frac{-\infty}{\infty} \end{aligned}$$



The background is a vibrant, abstract collage. It features a large, dark blue, stylized letter 'L' on the left. To the right, there are various mathematical elements: a large yellow sphere, a blue sphere, and a purple sphere. In the top right corner, there is a complex mathematical expression involving a square root of 5, a fraction with 1 and sqrt(5) in the numerator and 2 in the denominator, and a power of 2. The overall color palette is dominated by purple, blue, and yellow, with a high-contrast, pixelated texture.

# Math 1552

## ***Section 8.8: Improper Integrals***

Math 1552 lecture slides adapted from the course materials  
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# Today's Learning Goals

- Be able to identify when an integral is improper *(three cases)*
- Rewrite an improper integral as a limit
- Understand the meaning of convergence and divergence as relating to integration
- Evaluate improper integrals



# Improper integrals

$$\int_a^b f(x) dx$$

A definite integral is improper if:

- The function has a vertical asymptote at  $x=a$ ,  $x=b$ , or at some point  $c$  in the interval  $(a,b)$ .
- One or both of the limits of integration are infinite (positive or negative infinity).

$$\int_0^{2\pi} \tan(x) dx$$

$\rightarrow \tan(\pi/2) = +\infty$   
 $\rightarrow \tan(3\pi/2) = -\infty$

$$\int_{-\infty}^{\infty} f(x) dx, \int_1^{\infty} g(x) dx$$



Which of the following integral(s) is (are) improper? Why / which case?

✓ 1)  $\int_0^{\frac{\pi}{4}} \tan(2x) dx$       $\tan\left(\frac{2 \cdot \pi}{4}\right) = +\infty$

2)  $\int_{-1}^1 \frac{x-3}{x^2-2x-3} dx$

$\hookrightarrow \lim_{x \rightarrow \frac{\pi}{4}} \tan(2x) = +\infty$

3)  $\int_0^{\frac{\pi}{2}} \cos(x) dx$

4)  $\int_0^3 \frac{x-2}{x^2-6x+8} dx$